

# Trigonometry - Angles and Ratios Lesson #5: Special Triangles and Exact Values

## Overview

In this lesson, we will determine the exact value of the sine ratio, the cosine ratio, and the tangent ratio for a given angle with a reference angle of  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$ .

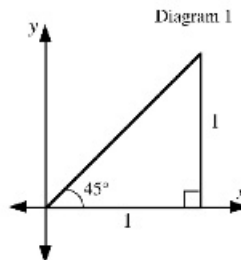
## Investigation

- a) Diagram 1 shows an angle of  $45^\circ$  in standard position. An isosceles triangle is drawn whose equal sides are 1 unit.

i) Determine the length of the hypotenuse.

ii) Use SOHCAHTOA or the  $x, y, r$  formulas to complete:

$$\sin 45^\circ = \quad \cos 45^\circ = \quad \tan 45^\circ =$$

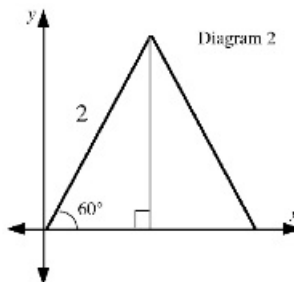


- b) Diagram 2 shows an angle of  $60^\circ$  in standard position. An equilateral triangle is drawn whose equal sides are 2 units, and a vertical altitude is drawn which divides the equilateral triangle into two congruent triangles.

i) Determine the length of the altitude.

ii) Complete :

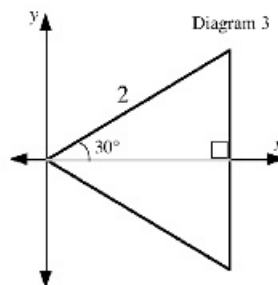
$$\sin 60^\circ = \quad \cos 60^\circ = \quad \tan 60^\circ =$$



- c) Diagram 3 shows an angle of  $30^\circ$  in standard position. An equilateral triangle is drawn whose equal sides are 2 units, and a horizontal altitude is drawn which divides the equilateral triangle into two congruent triangles.

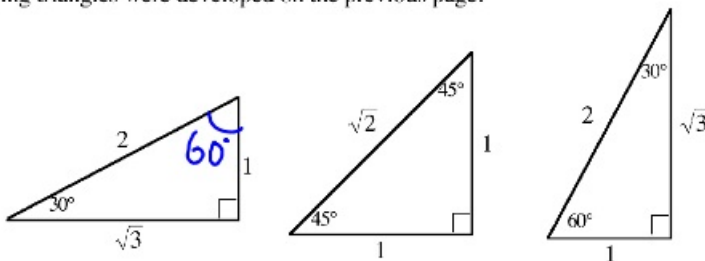
i) Complete:

$$\sin 30^\circ = \quad \cos 30^\circ = \quad \tan 30^\circ =$$

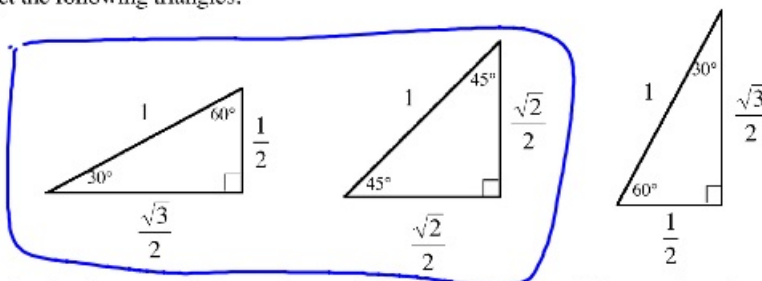


**Special Triangles**

The following triangles were developed on the previous page.



If we consider similar triangles to the above, all with hypotenuse length of one unit, we get the following triangles.



The triangles above are similar to the ones in the investigation and illustrate the trigonometric ratios as exact values for angles of 30°, 45° and 60°.

In each diagram, the horizontal distance is  $x$ , the vertical distance is  $y$  and the hypotenuse is  $r = 1$ .

Complete the following table.

$x^\circ$	30°	45°	60°
$\sin x$	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos x$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$
$\tan x$	$1/\sqrt{3}$	1	$\sqrt{3}$

**Finding Exact Trigonometric Ratios for Angles of 0° and 90°**

a) Consider a rotation angle of 0°. In this case  $x = 1, y = 0$  and  $r = 1$ .

b) Consider a rotation angle of 90°. In this case  $x = 0, y = 1$  and  $r = 1$ .

$y/r$	$\sin 0^\circ$	$0/1 = 0$		$\sin 90^\circ$	$1/1 = 1$	
$x/r$	$\cos 0^\circ$	$1/1 = 1$		$\cos 90^\circ$	$0/1 = 0$	
$y/x$	$\tan 0^\circ$	$0/1 = 0$		$\tan 90^\circ$	$1/0 = \text{undefined}$	

c) Explain why  $\tan 90^\circ$  is undefined.

### Determining Exact Values for Trigonometric Ratios of Certain Angles

There are several ways to determine, without technology, the exact value of the sine, cosine and tangent ratios of a given angle with a reference angle of  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$ .

We will discuss two of these methods:

- by reference angle and chart
- by the unit circle (in the next lesson)

### Using a Chart for Trigonometric Ratios of Special Triangles

We can summarize the exact values of trigonometric ratios of  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$  in the following chart.

$x$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined



**This chart should be memorized.**

Note the following patterns:

- The sine ratios increase from 0 to 1. The cosine ratios decrease from 1 to 0.
- The tangent ratios are equal to the sine ratios divided by the cosine ratios.

### Determining Exact Values of Trigonometric Ratios Using the Chart

We can use the previous table, together with the concept of reference angles and the CAST rule, to determine the exact values of the trigonometric ratios of certain angles in quadrants 2, 3, and 4.

Class Ex. #1



Write the following in terms of a reference angle and determine the exact value.

a)  $\sin 210^\circ$

sin -ve in Q3  
ref  $\angle = 30^\circ$

$$\sin 210^\circ = -\sin 30^\circ \\ = -\frac{1}{2}$$

b)  $\cos 300^\circ$

cos +ve in Q4  
ref  $\angle = 60^\circ$

$$\cos 300^\circ = \cos 60^\circ \\ = \frac{1}{2}$$

c)  $\tan 225^\circ$

tan +ve in Q3  
ref  $\angle = 45^\circ$

$$\tan 225^\circ = \tan 45^\circ \\ = 1$$

Class Ex. #2

Without using technology, determine the exact value(s) of  $\theta$  where

a)  $\cos \theta = -\frac{\sqrt{3}}{2}$ ,  $0^\circ \leq \theta \leq 360^\circ$

b)  $\tan \theta$  is undefined,  $0^\circ \leq \theta \leq 360^\circ$

cos-ve in Q2, Q3

ref  $\angle = 90^\circ$

ref  $\angle = 30^\circ$

$\theta = 90^\circ$  in Q1/2

$\theta = 180 - 30 = 150^\circ$  in Q2

$270^\circ$  in Q3/4

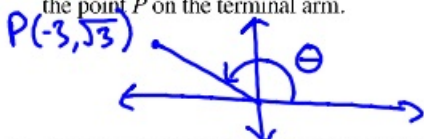
$= 180 + 30 = 210^\circ$  in Q3

Class Ex. #3

The point  $P(-3, \sqrt{3})$  is on the terminal arm of an angle  $\theta$ .

Without using technology, complete the following questions.

- a) Draw the angle in standard position and mark the point
- $P$
- on the terminal arm.



- b) State the values of
- $x$
- and
- $y$
- and hence the value of
- $\tan \theta$
- .

$x = -3$ ,  $y = \sqrt{3}$

$\tan \theta = -\frac{\sqrt{3}}{3}$

- c) State the reference angle and hence the rotation angle
- $\theta$
- .

ref  $\angle = 30^\circ$

$\theta = 180 - 30 = 150^\circ$  in Q2

Do #1-6

Memorize special triangles and/or table

## Complete Assignment Questions #1 - #8